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NUMERICAL ANALYSIS OF AXISYMMETRIC BUCKLING OF A CONICAL SHELL UNDER RADIAL COMPRESSION

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The nonlinear boundary-value problem of the axisymmetric buckling of a simply supported conical shell (dome) under a radial compressive load applied to the supported edge is formulated for a system of six first-order ordinary differential equations for independent fields of finite displacements and rotations. Multivalued solutions are obtained using the shooting method with specified accuracy. For various values of the loading parameter, bifurcation of the solutions of the problem is studied and a parametric branching diagram is constructed. The buckling modes are obtained for three branches of the solution. Curves of the buckling modes corresponding to three isolated branches of the solution are given.

Key words: *conical shell, buckling, numerical analysis.*

The nonlinear problem of the axisymmetric buckling of conical shells under a uniform pressure was studied in [1]. The axisymmetric buckling of plates under radial compression was analyzed in [2, 3]. These papers contain references to the earlier publications addressing these problems. To the author's knowledge, no results on the buckling of radially compressed shells (domes) are available in the literature.

System of Equations. In cylindrical coordinates (r, φ, z), the meridian of a conical undeformed dome is defined by the parametric equations

$$r = bt, \qquad z = a(1 - t) \qquad \forall t \in [0, 1],$$

where a is the height of the dome, b is the radius of the supported contour, and t is an independent variable reckoned along the meridian. We study deformation of the dome such that its midsurface remains axisymmetric. In the actual state, the meridian is defined by the parametric relations

$$r = ly_2(t), \qquad z = ly_3(t) \qquad \forall t \in [0, 1],$$

where l is the length of the meridian and $y_2(t)$ and $y_3(t)$ are unknown coordinates of a point. The parameters a, b, and l are related by the formulas $a = l\sin\alpha$ and $b = l\cos\alpha$, in which α is the slope of the meridian to the base plane.

To analyze the axisymmetric deformation of the dome, we use the equations of the nonlinear shell model with independent fields of finite displacements and rotations [1]. In addition to the functions $y_2(t)$ and $y_3(t)$, which define the point position, we introduce the function $\theta(t)$ — the angular displacement of the local coordinate basis relative to the cylindrical basis. The material of the shell is assumed to be transversely isotropic and linear elastic.

The system of equations formulated in [1] is valid for an arbitrary shell of revolution. In the case considered, we use the system in modified form

$$y_0' = x^{-1}[(1-\nu^2)y_1 - \nu(\sin y_0 - \sin\alpha)], \quad y_1' = x^{-1}(\nu y_1 + \sin y_0 - \sin\alpha)\cos y_0 + \varepsilon^{-1}f_3 - xq_2,$$

$$y_2' = \varepsilon\gamma x^{-1}f_3\sin y_0 + (1+\varepsilon f_1)\cos y_0, \qquad y_3' = \varepsilon\gamma x^{-1}f_3\cos y_0 - (1+\varepsilon f_1)\sin y_0,$$

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$$y_4' = x^{-1}[\nu f_2 + \varepsilon^{-1}(y_2 - x)] - xp_1, \qquad y_5' = -xp_3, \tag{1}$$

$$f_1 \equiv x^{-1}[(1-\nu^2)f_2 - \varepsilon^{-1}\nu(y_2 - x)], \qquad f_2 \equiv y_4 \cos y_0 - y_5 \sin y_0,$$

$$f_3 \equiv y_4 \sin y_0 + y_5 \cos y_0, \qquad x \equiv t \cos\alpha$$

with six unknown functions

$$y_0 = \theta, \qquad y_1 = \frac{xM_{11}l}{H}, \qquad y_2 = \frac{r}{l}, \qquad y_3 = \frac{z}{l}, \qquad y_4 = \frac{xT_1}{C}, \qquad y_5 = \frac{xT_3}{C}$$

and parameters α, $\gamma = E/G$, $\varepsilon^2 = h^2/(3l^2)$, $p_j = P_j l/C$, and $q_2 = Q_2 l^2/H$. Here the following notation is introduced: $C = 2h\varepsilon E$, $3H = 2h^3 E$, E is Young's modulus, G is the shear modulus, ν is Poisson's ratio, $T_{ii}(t)$ and $M_{ii}(t)$ are the components of the forces and moments in the local basis, respectively, $T_1(t)$, $T_3(t)$, $P_1(t)$, $P_3(t)$, and $Q_2(t)$ are the components of the tractions, surface forces, and moments in the cylindrical basis, $i = 1, 2$, $j = 1, 3$, and the prime denotes differentiation with respect to t.

The quasilinear system (1) defines the axisymmetric states equilibrium of a conical dome for specified values of the loading parameters p_1, p_3, and q_2, the stiffness parameters ε, γ, and ν, and specified boundary conditions for the supported contour. The system is singular for the variable t at the point $t = 0$. For $\gamma = 0$, system (1) becomes a system of equations of the problem of strong bending of a dome without transverse shear strains (the Kirchhoff model for large rotations). The plate buckling problem follows from (1) for $\alpha = 0$.

For system (1), we formulate the boundary-value problem of axisymmetric buckling of a conical dome compressed by uniformly distributed load of intensity P at the edge. Since no surface loads are applied, we set $q_2 = p_3 = p_1 = 0$ in (1). At the pole $t = 0$, the symmetry and regularity conditions should be satisfied. In the numerical analysis, these conditions are replaced by the following asymptotically accurate conditions at the point $t = \delta$ close to the pole (see [1, 3]):

$$\begin{gathered} y_5 = 0, \qquad (\nu - 1)y_1 + \sin y_0 - \sin\alpha = 0, \\ (\nu - 1)(y_4 \cos y_0 - y_5 \sin y_0) + \varepsilon^{-1}(y_2 - \delta) = 0. \end{gathered} \tag{2}$$

These relations ensure that the equalities $T_3 = 0$, $T_{22} = T_{11}$, and $M_{22} = M_{11}$, which should hold at the pole, are satisfied at the above-mentioned point.

At the boundary point $t = 1$, the conditions of a movable simply supported contour are specified

$$y_1(1) = 0, \qquad y_3(1) = 0, \qquad y_4(1) = -p \qquad (p = P/C). \tag{3}$$

Equations (1) and conditions (2) imply the obvious result $y_5(t) \equiv 0$ (i.e., $T_3 \equiv 0$), which was not used in the numerical algorithm and served to verify the stability of the solutions against small perturbations of the boundary parameters.

The nonlinear boundary-value problem (1)–(3) was solved by the method of shooting from the point $t = 1$ to the point $t = \delta$. Boundary conditions (2) lead to a system of implicit equations for the additional initial parameters of the shooting method. Bifurcation of the solutions of the boundary-value problem was studied with variation in the loading parameter. For numerical implementation of the algorithm, the Mathcad software package was used.

Numerical Results. Some results of solution of the nonlinear boundary-value problem (1)–(3) for isotropic conical domes are represented in tables and figures. In the calculations, the parameter α was varied and the other parameters were fixed: $\nu = 0.25$, $\gamma = 2.5$, and $\varepsilon = 0.025$. We introduce the following notation: $\vartheta = \theta(1)$ is the boundary value of the angular displacement, $u = 1 - r(1)/b$ is the parameter of the radial displacement of the supported contour, $w = z(\delta)/a + \delta - 1$ is the parameter of the axial displacement of the point $t = \delta$, $\tau_i = T_{ii}/C$ are parameters describing the internal forces, $\mu_i = M_{ii}R/H$ are parameters describing the internal moments, $\tau_0 = \tau_1(\delta) = \tau_2(\delta)$, and $\mu_0 = \mu_1(\delta) = \mu_2(\delta)$.

In accordance with the recommendations of [2], the dependence of the solutions on the loading parameter $p = P/C$ is represented by diagrams on the plane (p, u), where u is the percentage ratio of the radial displacement to the radius of the supported contour b ($u > 0$ if the supported point is displaced toward the pole).

Figure 1 compares the branching of the solutions for a dome with angle $\alpha = \pi/180$ with that for a plate. Solid curves 1, 2, and 3 are the branches (trajectories) of buckling modes of the dome. They are separated by the branches of buckling modes of the plate (dashed curves) [2]. The branches of the dome are enclosed between the

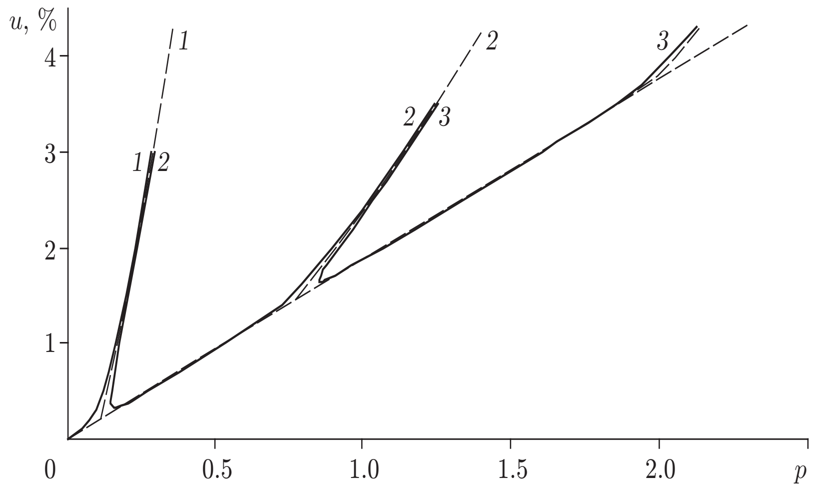

Fig. 1

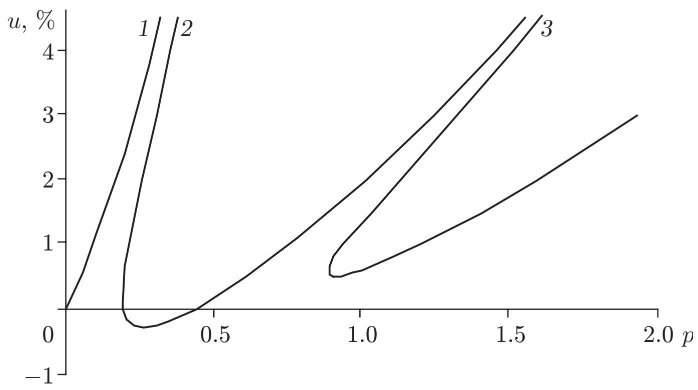

Fig. 2

branches of buckling modes of the plate and the branch of its plane (fundamental) modes of equilibrium (the lower dashed curve issuing from the coordinate origin). The buckling modes corresponding to the first branch issuing from the coordinate origin (the point of the initial state of the dome) should be considered the fundamental modes of the dome. Figure 1 shows only the first three branches of buckling modes of the dome and the plate.

As the angle α increases, the branches of higher modes of the dome descend below the line that represents the plane modes of the plate and even below the abscissa axis. Figure 2 shows these solutions for a dome with angle $\alpha = \pi/12$ (15-degree dome). In this case, the second branch of buckling modes intersects the abscissa, which implies that for the compressive radial force ($p > 0$) there are buckling modes with zero and negative values of the parameter u. In the latter cases, the supported contour is extended rather than compressed. This result seems paradoxical but it is valid and will be discussed below.

The branching of the deformation path for a 15-degree dome is shown diagrammatically in Fig. 3. It is depicted in the plane (p, w), where w is the conventional parameter of the axial displacement normalized by the dome height a. Although this diagram differs radically from that shown in Fig. 2, there is one-to-one correspondence between the branches shown in Figs. 2 and 3. Therefore, any of these diagrams serves to illustrate the branching of the solutions of the problem considered. However, Fig. 2 is more informative since its coordinate parameters are energetically conjugate quantities.

Numerical values of several parameters of states for a 15-degree dome at separate points of branches 1, 2, and 3 (Fig. 2) are listed in Tables 1, 2, and 3, respectively. One of the parameters was specified in the calculations,

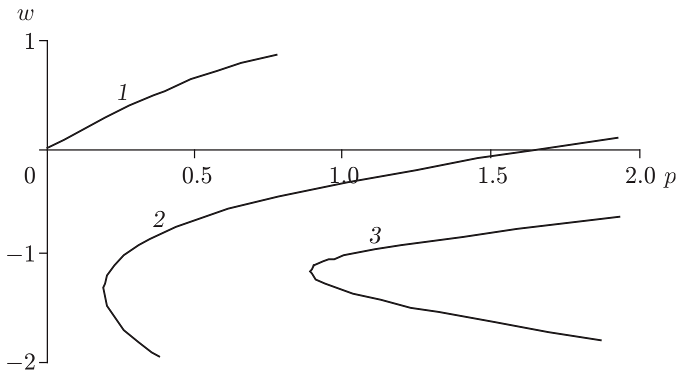

Fig. 3

TABLE 1

Point No.	p	$100u$	w	ϑ	$\tau_1(1)$	$\tau_2(1)$	τ_0	μ_0
1	0.1054	1.150	0.1529	0.1	−0.0986	−0.4845	0.0511	−0.0189
2	0.1955	2.411	0.2825	0.2	−0.1750	−1.0082	0.0892	−0.0495
3	0.2783	3.786	0.3953	0.3	−0.2355	−1.5732	0.1191	−0.0844

TABLE 2

Point No.	p	$100u$	w	ϑ	$\tau_1(1)$	$\tau_2(1)$	τ_0	μ_0
1	0.3064	3	−1.8108	−0.7791	−0.2663	−1.2666	−0.9138	−0.9958
2	0.2123	1	−1.5448	−0.6275	−0.1982	−0.4496	−0.9434	−0.6730
3	0.1919	0	−1.2725	−0.5400	−0.1845	−0.0461	−0.8322	−0.2037
4	0.2573	−0.28	−1.0139	−0.5005	−0.2500	0.0495	−0.5903	0.2183
5	0.4385	0	−0.7372	−0.4834	−0.4278	−0.1069	−0.2817	0.5029
6	0.7608	1	−0.4684	−0.4977	−0.7397	−0.5849	0.0102	0.6178
7	1.2466	3	−0.1920	−0.5542	−1.1937	−1.4984	0.2688	0.5846
8	1.8348	6	0.06558	−0.6424	−1.7035	−2.8259	0.4533	0.4562

TABLE 3

Point No.	p	$100u$	w	ϑ	$\tau_1(1)$	$\tau_2(1)$	τ_0	μ_0
1	1.6958	5	−1.7147	0.08205	−1.5965	−2.3991	−1.1273	−2.7789
2	1.3246	3	−1.5391	0.01522	−1.2741	−1.5185	−1.3489	−2.6142
3	0.9058	0.51	−1.1072	−0.1164	−0.8962	−0.4281	−1.7646	−1.0081
4	1.2004	1	−0.9156	−0.1254	−1.1892	−0.6973	−1.4630	0.3990
5	1.9287	3	−0.6451	−0.0677	−1.8925	−1.6731	−0.4959	1.5980
6	2.5359	5	−0.4580	−0.0138	−2.4583	−2.6146	0.0775	1,7549

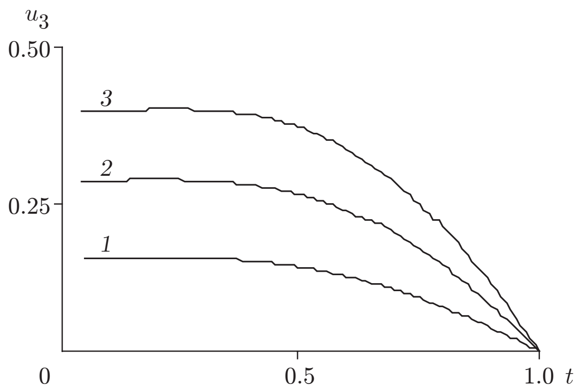

Fig. 4

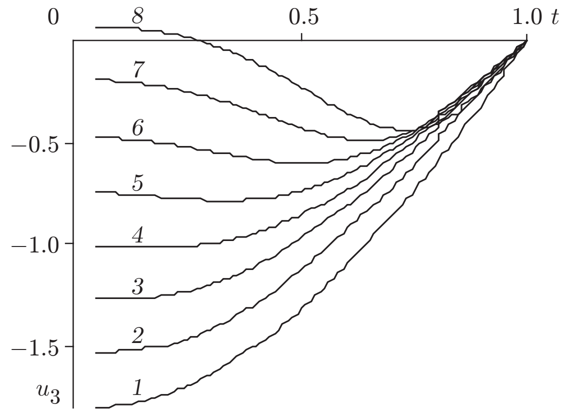

Fig. 5

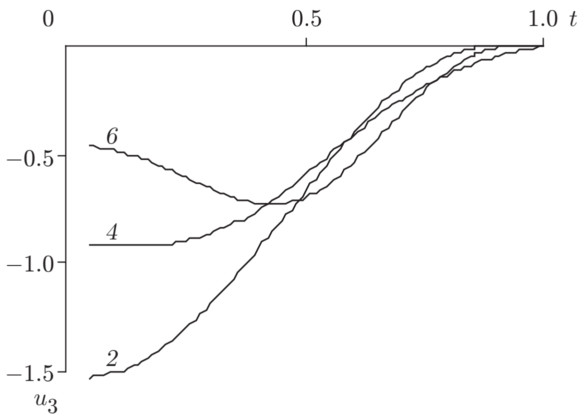

Fig. 6

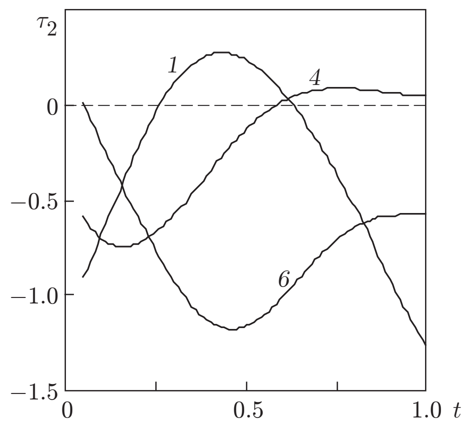

Fig. 7

and the other parameters were determined by the shooting method. These data gives insight into the evolution of the parameters of states of the dome. For the states of equilibrium corresponding to the first branch, all parameters vary monotonically as the load increases (see Table 1). The monotonicity is violated for the states corresponding to the higher-level branches. A distinguishing feature of these states is the presence of alternating-sign parameters. Attention is drawn to point 4 (see Table 2) which shows that the parameters u and $\tau_2(1)$ change sign simultaneously.

The buckling modes of a 15-degree dome corresponding to some points of branches 1, 2, and 3 are given in Figs. 4, 5, and 6, respectively, in the form of curves of the axial displacement parameter (deflection) $u_3 = z/a + t - 1$ versus the meridian coordinate t. Curves 1, 2, and 3 in Fig. 4 correspond to the points of the first branch given in Table 1. They are similar to the first buckling modes of the plate [3]. Figure 5 shows the deflections of the meridian for the second branch (for the data in Table 2); curves 3 and 5 are two buckling modes corresponding to the value $u = 0$ and curve 4 is the buckling mode for a negative value of the parameter u. Figure 6 illustrates the deflections of the meridian for the third branch (for the data given in Table 3). One can see that the deflection curve of the first branch has no points of inflection and the deflection curves of the second and third branches have one and two points of inflection, respectively. The dome height increases for positive deflections and decreases for negative deflections (to the state where the dome is turned inside out for $w = -2$).

Figure 7 shows the evolution of the hoop stresses for the buckled states of the second branch in transition from point 1 to point 6 in accordance with Table 2. The tensile-stress zone occurring in a ring region inside the dome (curve 1) is gradually extended toward the edge (curve 4), which results in its extension. Beyond point 4, the tensile-stress zone near the edge vanishes and a compression zone occurs (curve 6).

Conclusions. Numerical solutions of the nonlinear problem formulated were obtained by the shooting method with specified accuracy (Figs. 4–7). The branched deformation paths of the dome are represented by plane projections (Fig. 1–3). In contrast to the plate, the deformation paths of the dome consist of isolated branches and a discontinuity occurs in the neighborhood of the bifurcation points of the plate for arbitrarily small deviations of the dome from the plate. Therefore, a sudden loss of stability of the dome is possible. Under compressive loads, equilibrium states of the dome with an extended edge contour were found.

The equations obtained allows one to formulate and solve strongly nonlinear axisymmetric problems of isotropic and transversely isotropic shells under other loads and boundary conditions.

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